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The Adequacy of the Traditional Econometric Approach to Non-linear Cycles

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resumo

Neste artigo usa-se uma extensão do modelo de ciclos de crescimento de Goodwin para criar dados artificiais que seguem um processo (determinístico) caótico. Usam-se esses dados para ilustrar a desadequação da econometria tradicional para lidar com series caóticas. Para descrever o processo gerador dos dados, estima-se um modelo autoregressivo. Apesar de alguns dos testes tradicionais não evidenciarem problemas de má-especificação do modelo, este evidencia propriedades qualitativas diferentes do verdadeiro processo gerador de dados. Apresenta-se um procedimento desenvolvido para lidar com a possibilidade de caos: a estatística BDS. Também se sugere uma justificação para a pouca evidência de caos em dados macroeconómicos agregados.

résumé / abstract

Pour montrer que l'approche économétrique traditionnelle ne peut pas traiter le chaos déterministique, j'emploie une extension du modèle de cycle de la croissance de Goodwin pour generer des données de production artificielles.Un modèle EGARCH est estimé décrire le procédé de génération des données. Bien qu' en utilisant quelques tests économétriques traditionnels, aucune évidence d' erreur de spécification ne soit trouvée, le processus estimé est qualitativement erroné: il est dynamiquement stable quand le veritable processus est en fait instable. Un procédé économétrique spécifique, développé pour traiter le chaos déterministe est présenté: les statistiques de BDS. En outre, une explication de l'absence d'évidence du chaos déterministique dans les series temporelles macroeconomiques agregées est avancée.

To show that the traditional econometric approach is not able to deal with deterministic chaos, I use an extension of Goodwin's growth cycle model to generate artificial data for output. An EGARCH model is estimated to describe the data generation process. Although, using some traditional econometric tests, no evidence of misspecification is found, the estimated process is qualitatively wrong: it is dynamically stable when the true process is unstable. A specific econometric procedure developed to deal with deterministic chaos is presented: the BDS statistics. Also an explanation for the little evidence of deterministic chaos in aggregated macroeconomic time series is suggested.

1. Introduction*

> "Thanks to the early work of Frisch (1933) and Slutsky (1937) (...) most macroeconomists now share the same general analytical approach, that based on the distinction between impulse and propagation mechanisms."
>
> — in *Blanchard and Fischer* (1989)

Slutsky's paper published in 1937 in Econometrica was a revised English version of an earlier paper (1927) published in Russia. The main objective was to explain "*the undulatory character of the processes and the approximate regularity of the waves*" using nothing but random series. Although Slutsky was not able to give a general law to explain what kind of cycles can be generated by a moving summation of random series, Slutsky was able to establish that some sort of swings could be produced by a simple aggregation of erratic influences.

Frisch was aware of Slutsky's results, while writing his 1933 article. Frisch constructed a model which, when disturbed, would generate damped oscillations (the propagation mechanism). To explain why the cycles are undamped and show some regularity, Frisch considered the impact of exogenous random shocks, which provided the necessary energy to feed the cycles. The basic idea was the rocking horse analogy of Wicksell: "if you hit a rocking-horse with a stick, the movement of the horse will be very different from that of the stick. The hits are the cause of the movement, but the system's own equilibrium laws condition the form of the movements"[1].

Authors who have not accepted this approach are innumerous (Schumpeter, Harrod, Goodwin, Kaldor, Kalecki, etc.). These economists had a different approach. They considered the economic system to be inherently unstable. In their opinion, even in the absence of exogenous shocks, the economic system fluctuates cyclically. This deterministic approach requires the consideration of a non-linear model. These ideas lost their influence in the 70's and 80's because the deterministic cycles had the unpleasant feature of being easily predictable, being incompatible with optimising rational agents. But, what if deterministic cycles are unpredictable? In that case the above argument no longer applies and the idea of deterministic non-linear models is resuscitated.

In the recent economic literature there is a number of examples of how classical models can be easily extended to accommodate the possibility of chaotic motions. The reader interested in a relatively recent survey may whish to consult Reichlin (1997).

A related problem, raised by some authors, e.g. Blatt (1983) and Louçã (1997), is to question the ability of current econometric procedures to deal with deterministic chaos.

In this paper, after a brief definition of deterministic chaos, an extension of Goodwin (1991)'s model is used to generate artificial data. I show that, with standard procedures, a linear (auto-regressive) econometric model, with no (apparent) evidence of misspecification, can be estimated. The properties of the estimated model are qualitatively wrong. Although Louçã (1997) did a similar exercise, there are two main differences between my approach and Louçã's approach: first, while Louçã took the evidence of heteroskedasticity as a sign of the existence of chaos, in this paper heteroskedasticity is modelled in a more modern (EG)ARCH model. Second, when "stationarizing" the data, before testing for non-linear structures, Louçã simply removed a linear trend. I perform a unit root test, and then take the first differences, as it is common econometric practice.

* The author is grateful to two anonymous referees, Francisco Louçã, and, especially, to António Alberto Santos, for helpful comments, suggestions, and corrections. The usual disclaimer applies.

1 Wicksell, K. (1918), Karl Petander: Goda och dårliga tider", *Ekonomisktidskrift*, vol. 19, pp. 66-75, cit. in Thalberg (1990).

I conclude then that to accommodate the possibility of deterministic chaos specific procedures have to be developed. One of them is presented: the BDS statistics, which was developed to detect low dimensional deterministic chaos (although it has power to detect other non-linearities as well). It is also illustrated a possible explanation for little evidence of deterministic chaos in macroeconomic time series (while in financial time series there is overwhelming evidence of non-linear dependence and some evidence of chaos): aggregation can hide evidence of chaos.

2. Chaos: Attractors and Sensitive Dependence on Initial Conditions

The technical definition of a chaotic motion lies heavily in three concepts:

— Conservative and Dissipative Systems,

— Sensitive Dependence on Initial Conditions (SDIC), and

— (Strange) Attractors.

A conservative system is a system without any kind of friction, thus it conserves its total volume (for example total energy) along the phase space. A *dissipative system*, see Medio (1992), is characterized by a contraction of phase space volumes with increasing time. Formally this can be defined as the Jacobian of the system of differential equations having a negative trace. Because of dissipation, an n-dimensional system will eventually become confined to a subset of Hausdorff dimension[2] less than n (which can be a fractal dimension).

Let $\phi_t : U \rightarrow U, U \subseteq R^n$, be the solution of a system of ordinary differential equations. The system shows *SDIC* if there exists $\delta > 0$ such that, for any $x \in U$ and any neighbourhood N of x, there exists $y \in N$ and $t \geq 0$ such that $|\phi_t(x) - \phi_t(y)| > \delta$, see Medio (1992). This definition says that even small differences in the initial conditions will lead to completely different paths, turning impossible to make long run forecasts.

Finally, we say that a compact set $A \subset U$, invariant under ϕ_t is a *strange attractor* for a system if there is a set B with the following properties:

1. B is an n-dimensional neighbourhood of A,
2. for any initial point $x(0) \in B$, the trajectory representing the solution $x(t)$ remains in B for all $t > 0$, and $\underset{t\rightarrow\infty}{x(t)} \rightarrow A$,
3. there is SDIC,
4. there is a dense orbit on A for the flow ϕ_t.

The meaning of the first three conditions is obvious. The last one means the attractor cannot be split into two regions. To have a strange attractor it is important that the system is a dissipative system; since it is dissipation that makes transitory phenomena disappear. Sensitivity to initial conditions is what makes chaotic systems so special. Because one can never characterize a system's initial state to infinite precision, it follows that long-term chaotic evolution can never be predicted. The best that one can do is to observe that there will be a probability distribution according to which neighbourhoods of an attractor are visited. Occasionally, this "invariant measure", can be calculated from a formula, but, in general, this is not possible.

2.1. The Rossler Attractor

One of the simplest systems that can generate chaotic motions is the Rossler band. Formally, the system of equations that I use in the next sections is a variant of the Rossler band. For more on this, the reader should consult Goodwin (1991). In this section I will just use this system to illustrate how chaotic series can be generated.

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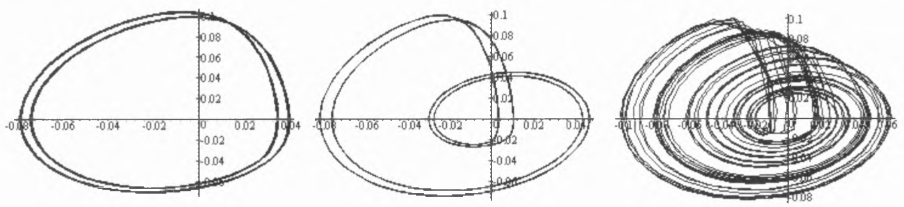
2 Let $N(\varepsilon)$ be the number of squares with length ε needed to cover an object, then the Hausdorff dimension is given by $D = \lim_{\varepsilon \rightarrow 0} \frac{\ln(N(\varepsilon))}{\ln(1/\varepsilon)}$.

Consider the following system:

$$\begin{cases} \mathbf{U}' = 0.5\mathbf{L} \\ \mathbf{L}' = -0.5\mathbf{U} + 0.15\mathbf{L} - 0.8\mathbf{Z} \\ \mathbf{Z}' = 0.001 + g\mathbf{Z}(\mathbf{L} - 0.048) \end{cases}$$

For small values of g we have a limit cycle of one period (not shown). But as g increases a bifurcation occurs, and the cycle period doubles. If we continue to increase g another bifurcation occurs, creating a limit cycle of four periods. For g high enough we have an infinite period limit cycle, i.e. chaos. See figure 1 to observe the evolution of the phase diagrams of **U** and **L** as g increases.

Figure 1: Period doubling and Chaos: The Rossler Band



two period cycle (g = 35) for period cycle (g = 65) Chaos – the Rossler attractor (g = 85)

3. A Model of Growth and Cycles

Goodwin (1991) extended his 1967 predator-prey model in order to accommodate growth and cycles. The model generated a Kondratieff growth cycle, which also incorporated Juglar cycles.

Goodwin incorporated a Schumpeterian swarm of innovations according to which, after a weak beginning, the path-breaking innovation proves its importance and more and more firms will adopt the innovation. At the end, the rate of adoption will diminish since the majority of the firms have already adopted it.

For convenience Goodwin's system of differential equations[3] is reproduced here (all variables should be interpreted as deviations from the steady state values, being possible to take positive and negative values):

$$\begin{cases} \mathbf{U}' = h\mathbf{L} \\ \mathbf{L}' = -d\mathbf{U} + f\mathbf{L} - e\mathbf{Z} \\ \mathbf{Z}' = b + g\mathbf{Z}(\mathbf{L} - c) \end{cases} \tag{1}$$

U represents the labour proportion of national income; **L** the rate of employment and **Z** can be interpreted as a control parameter, e.g. government budget surplus. As illustrated in the previous section for sufficiently high values of g the system generates deterministic chaos. The first equation of the system simply says that the labour proportion of national income increases with the rate of employment. The second equation, says that the evolution of the rate of employment depends positively on the rate of employment and negatively on the labour's share of income and on the control parameter **Z**. As it can be seen the only (simple) non-linearity is introduced in the third equation, the one describing the dynamics of the control parameter **Z**. It is interesting to note how such a simple non-linearity is responsible for the reach erratic dynamics that we will observe in the rest of the paper.

3 For details on this, and in the next equations the reader is invited read the cited works of Louçã and Goodwin.

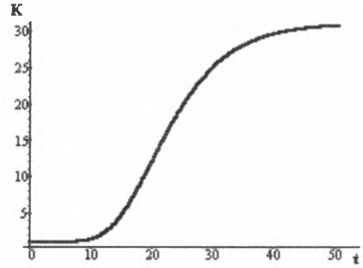
To have an endogenous growth component in this model I admit that cyclical labour productivity is influenced by investment (more specifically the cyclical component of labour productivity is considered to be $\gamma \frac{\mathbf{K'}}{\mathbf{K}}$, where K stands for the stock of capital).

For investment I admit an exogenous Shumpeterian swarm of innovations[4]:

$$\mathbf{K'} = m \exp\left(n - qt - \exp\left(n - qt\right)\right) \tag{2}$$

Calibrating this equation with the values m = 4.5, n = 3, q = 0.15 and $K_0 = 1$ (where K_0 stands for the initial stock of capital) we have the dynamics described in figure 2 (t stands for time).

Figure 2: Capital Evolution for 50 years



The dynamics of output is determined by the evolution of total employment (= L* + L[5]) and by the evolution of labour productivity $\gamma \frac{\mathbf{K'}}{\mathbf{K}}$:

$$\frac{\mathbf{Y'}}{\mathbf{Y}} = \frac{\mathbf{L'}}{\mathbf{L^*} + \mathbf{L}} + \gamma \frac{\mathbf{K'}}{\mathbf{K}} \tag{3}$$

where Y stands for output.

To consider a more general model, instead of equation (2) we will admit an accelerator mechanism. I will include directly the employment level in the investment equation. So equation (2) becomes:

$$K' = m \exp\left(n - L - qt - \exp\left(n - L - qt\right)\right) \tag{4}$$

Joining all the equations, the complete model becomes:

$$\begin{cases} \mathbf{U'} = h\mathbf{L} \\ \mathbf{L'} = -d\mathbf{U} + f\mathbf{L} - e\mathbf{Z} \\ \mathbf{Z'} = b + g\mathbf{Z}\,(\mathbf{L} - c) \\ \mathbf{K'} = m \exp\left(n - \mathbf{L} - q\mathbf{t} - \exp\left(n - \mathbf{L} - q\mathbf{t}\right)\right) \\ \dfrac{\mathbf{Y'}}{\mathbf{Y}} = \dfrac{\mathbf{L'}}{\mathbf{L^*} + \mathbf{L}} + \gamma \dfrac{\mathbf{K'}}{\mathbf{K}} \end{cases} \tag{5}$$

To understand the kind of output dynamics generated by this model we calibrate it with the following parameter values: b = 0.001; c = 0.048; d = 0.5; e = 0.8; f = 0.15; g = 85; h = 0.5; L* = 0.9; n = 3; q = 0.15; γ = 0.3. One hundred years of data is generated. Since the

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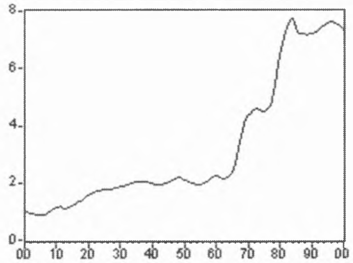
4 This function is known as the Gompertz curve and it is a special case of the generalised logistic. If I had used a simple logistic the main implication would have been a greater variation of the series accumulated in early stages. Cadima (1999) used a similar function to model the evolution of the cellular phones in Portugal.

5 L* is the steady state value of the rate of unemployment, and L is, as mentioned before, the deviation from the steady state.

parameters chosen to the investment equation imply a swarm of fifty years I have to introduce a second swarm of fifty years. This way in the first fifty years m takes the value m = 4.5. Beyond that m takes the value m = 135.[6]

Figure 3: Output Time Evolution



The initial values chosen to generate the first 50 years of data were $U_0 = 0.02$, $L_0 = 0.04$, $Z_0 = 0$, $K_0 = 1$, $Y_0 = 1$. After generating the first fifty years of data, I repeat the procedure to generate the second fifty years. With the obvious difference that now I will consider m = 135, and the initial values of the second series are the terminal values of the first series.

The results can be observed in figure 3. One interesting feature of the time series generated is that the cycles generated are not identical, even considering identical capital accumulation dynamics for both half centuries. We observe that a chaotic deterministic system can generate a quite erratic behaviour. The possibility that an erratic behaviour can be purely deterministic raises an interesting question: to what extent are the traditional econometric techniques appropriate to deal with this new issue? I try to sketch the answer in the next section.

4. An Econometric Application to our Artificial Model

Blatt (1983) alerted to the dangerous consequences of an error in the identification of the stability properties of an economic system, while questioning the ability of the traditional econometric tools to analyse the stability of an economic system. To answer this question Blatt made a simple test. Blatt generated some economic time series with the help of a non-linear, locally unstable, macro-model proposed by Hicks. With the artificial data Blatt tried to estimate the original model. The results were quite unpleasant: the estimated model did not identify the inherent instability of the original model. Basically, a dynamically stable model was estimated and the endogenous cycles were attributed to stochastic shocks, with no statistic evidence of misspecification.

Louçã (1997) took a similar approach. Louçã considered a more general model to generate artificial data for output that was able to simulate growth and cycles endogenously[7]. But, in the treatment of the time series output, a linear trend[8] was extracted and then the residuals were modelled as a linear autoregressive process. The problem with Louçã's approach is that when one tries to apply usual econometric procedures to time series data it is good practice to test

6 The choice of m = 4.5 was arbitrary. The choice of m = 135 was to guarantee that the evolution of capital would be the same in both periods. Basically, using equation 2, m = 4.5 implies that capital is 1 in period 0 and 30.67 after 50 years. Thus the initial value was multiplied by 30. In the second swarm of innovation the initial capital is 30.67 and, after 50 years, becomes 920 (so the initial value was multiplied by 30 again). This way the evolution of capital is essentially the same in both periods. If I had presented the graph of the evolution for m = 135 and initial capital of 30.67, the graph would be identical to figure 1, with the obvious exception of the scale of the vertical axes.

7 The model used was very similar to the system of equation 5. The main difference is that Louçã represented the investment dynamics with a simple logistic and did not introduce an accelerator component in the investment function.

8 The trend was extracted from logaritmized time series, so it is an exponential trend relative to the original data.

previously the stationarity of the series. Before extracting a (linear) time trend, it is necessary to test the possibility of the series being difference stationary and not trend stationary.

I tested the stationarity of time series represented in figure 3 (the test used was the ADF test and was applied after logarithmizing the series). According to the test results, we could not reject the null hypothesis of non-stationarity for the log of the output (LY), while for the growth rate (DLY[9]) we reject the null hypothesis accepting the growth rate to be stationary around a constant. So an applied econometrist would not extract a linear trend to stationarize the series. He would rather consider the growth rate of Y.

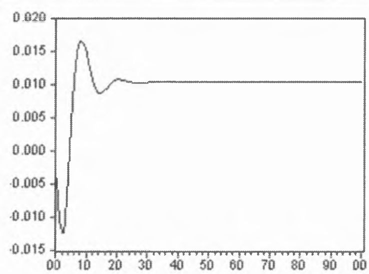
Given these results I will model DLY as an autoregressive process. Using Ordinary Least Squares to estimate the results we get:

$$DLY_t = \underset{(2.8)}{0.001} + \underset{(29)}{1.957}\, DLY_{t-1} - \underset{(-11.5)}{1.624}\, DLY_{t-2} + \underset{(6.7)}{0.939}\, DLY_{t-3} - \underset{(-5.3)}{0.355}\, DLY_{t-4} \quad (6)$$ [10]

where the values in parenthesis are the t-statistics. The R-squared (and also the adjusted R-squared) is about 96%. The residuals show no evidence of serial correlation[11]. The number of lags chosen was based on the Akaike and Schwarz information criteria. This choice is strengthened by the fact that higher lags are not statistically significant[12].

This estimated model fits perfectly in the Slutsky-Frisch's paradigm: there is an exogenous trend and two different growth cycles (one with 4.5 years and the other with 24.8 years) aggregated additively. In figure 4 we can see how this estimated model would work in the absence of stochastic shocks.

Figure 4: The estimated model dynamics



To explain the persistence of cycles in this model, Frisch (1965) would suggest the addition of a stream of exogenous shocks. This is what I do next. A stream of exogenous shocks with mean zero and variance 0.0001444 is added to equation 6 with the help of a normal random number generator[13]. In figure 5 we can compare the original artificial time series with the time series generated by the estimated model (augmented with the stochastic shocks)[14].

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9 DLY is the first difference of LY. Since LY = ln Y, DLY is the growth rate of Y.

10 Since I used semi-annual data, we have the growth rate of period t depending on the previous four semesters.

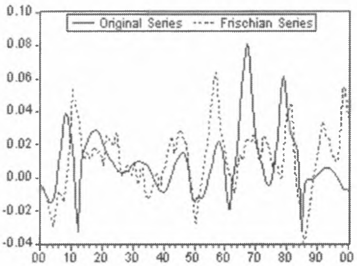
11 I used the Breusch-Godfrey Serial Correlation LM test. I tested for autocorrelation of the first, second, third, and fourth order. The P-values of the chi-square statistics were, respectively, 0.41, 0.15, 0.28, and 0.38.

12 For example, if we introduced a fifth lag, its P-value of the t statistic would be 0.39.

13 The value 0.0001444 was chosen to match the variance of this new time series with the original time series. Equation 6 becomes $DLY_t = 0.001 + 1.957\, DLY_{t-1} - 1.624\, DLY_{t-2} + 0.939\, DLY_{t-3} - 0.355\, DLY_{t-4} + \varepsilon_t$, where ε_t is the stochastic shock, generated using the Normal Number Random Generator of Excel.

14 Between applied macroeconomists it is also common to stationarize the output time series, using the Hodrick-Prescott filter, instead of taking the first differences. I did not choose this method because it is, in my opinion, an arbitrary method without a sound theoretical justification. Anyway, if that method was used the results would be similar to the ones presented in equation (6). These results are available at request to the author.

Figure 5: Two simulated Time Series



The econometric insufficiencies become apparent with these pictures. It is impossible to look to a linear model as a linear approximation of the true (non-linear) model when the true model is not locally stable. And that is what happens when one estimates a linear model, and the true data generation process is non-linear (and unstable). The estimated model will be stable (generating damped oscillations), and the persistence of the economic cycles will be left unexplained. This insufficiency is a fundamental one. Since the motion of a chaotic series is confined to a closed region (the strange attractor), it is impossible for a linear model to capture the local instability of the system. Any oscillating unstable linear model would generate explosive cycles, which could not be confined to a compact set.

4.1. Problems with heteroskedasticity

We have so far neglected the possibility of having heteroskedastic disturbances. In traditional time series analysis it was usual to consider homoskedastic processes (associating heteroskedasticity to cross-sectional data). But, at least since Engle (1982), one cannot put aside the possibility of having an Autoregressive Conditional Heteroskedasticity (ARCH) model or one of its extensions, as we shall see.

Consider a p^{th} order ARCH process:

$$\begin{cases} Y_t = \beta' X_t + \varepsilon_t \\ \varepsilon_t = u_t \sqrt{\varpi + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2} \end{cases} \tag{7}$$

where u_t follows a standard normal. It easy to derive the conditional and unconditional variances of ε_t:

$$\begin{cases} Var(\varepsilon_t | \varepsilon_{t-1}, \ldots, \varepsilon_{t-p}) = \sigma_t^2 = \varpi + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 \\ Var(\varepsilon_t) = \dfrac{\varpi}{1 - \sum_{i=1}^{p} \alpha_i} \end{cases} \tag{8}$$

In this situation, although the OLS estimator is the best linear unbiased estimator (BLUE), but there is a more efficient non-linear estimator. Engle (1982) derived the likelihood function for this model and also presented a Lagrange Multiplier (LM) test for the ARCH process.

Table 1: ARCH LM Test		
	obs × R^2	P-value
1st order	2.979	0.084
2nd order	15.199	0.001
3rd order	15.149	0.002
4th order	16.030	0.003

In table 1 we can see the results of the ARCH LM test. With these results an applied econometrist would have to deal with the conditional heteroskedasticity problem. In this work I considered first a Generalized ARCH (GARCH) model proposed by Bollerslev (1986) and also Taylor (1986). The advantage of this approach is that it is usually more parsimonious with the number of lags needed. In a GARCH (p,q) model the conditional variance is given by:

$$\sigma_t^2 = \varpi + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{9}$$

Bollerslev et al. (1994) show that this is equivalent to saying that ε_t^2 can be modelled as an $ARMA\left[\max(p,q),p\right]$ model. If equation 9 is correctly specified the standardized residuals should not exhibit additional ARCH.

After considering several GARCH models of different orders we concluded that the standardized residuals continued to exhibit ARCH, indicating that equation 9 was misspecified.

Nelson (1991) proposed an Exponential GARCH (EGARCH) model. Equation 9 is replaced by:

$$\ln(\sigma_t^2) = \varpi + \sum_{i=1}^{q} \left(\alpha_i \left| \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right| + \gamma_i \frac{\varepsilon_{t-i}}{\sigma_{t-i}} \right) + \sum_{j=1}^{p} \beta_j \ln(\sigma_{t-j}^2) \tag{10}$$

After re-estimating equation 6, admitting that the conditional heteroskedasticity follows an EGARCH (2,4)[15], we get:

$$DLY_t = \underset{(19.8)}{0.0003} + \underset{(281.7)}{3.10}\, DLY_{t-1} - \underset{(-135.4)}{3.61}\, DLY_{t-2} + \underset{(84.6)}{1.87}\, DLY_{t-3} - \underset{(-55.8)}{0.368}\, DLY_{t-4} \tag{11}$$

$$\ln(\sigma_t^2) = \begin{array}{l} -\underset{(-15.0)}{7.2} + \underset{(17.4)}{2.13}\left|\dfrac{\varepsilon_{t-1}}{\sigma_{t-1}}\right| - \underset{(-0.82)}{0.07}\dfrac{\varepsilon_{t-1}}{\sigma_{t-1}} + \underset{(9.2)}{1.81}\left|\dfrac{\varepsilon_{t-2}}{\sigma_{t-2}}\right| - \underset{(-0.26)}{0.02}\dfrac{\varepsilon_{t-2}}{\sigma_{t-2}} \\ + \underset{(0.9)}{0.06}\ln(\sigma_{t-1}^2) + \underset{(36.4)}{1.33}\ln(\sigma_{t-2}^2) - \underset{(-2.6)}{0.12}\ln(\sigma_{t-3}^2) - \underset{(-16.6)}{0.53}\ln(\sigma_{t-4}^2) \end{array} \tag{12}$$

where the values in parenthesis are the z-statistics. As it can be seen, the results of equation 11 do not differ substantially from the results obtained in equation 6. It is easy to verify that the stability properties do not change. When the ARCH LM test is applied to the standard residuals, the results are conclusive. As we can see in table 2 the null hypothesis of conditional homoskedastic residuals is not rejected. Even the Kiefer-Salmon test (also known as Jarque-Bera) normality test tends to accept the good specification of the model (the Kiefer-Salmon statistic has a value of 2.4 with a P-value of 0.3). Thus, not even the normality of the standard residuals would be rejected.

15 The order of the ARCH process was chosen, basically, with the help of the Akaike and Schwartz information criterion, and with significance tests.

Table 2: ARCH LM Test to the EGARCH standard residuals

	obs $\times R^2$	P-value
1st order	0.009	0.924
2nd order	1.693	0.429
3rd order	1.936	0.586
4th order	2.096	0.718

Although without performing a battery of tests, we can see a tendency to accept this wrong model. Wrong because equation 11 represents a stable model, when we know the true underlying process is a nonlinear unstable one. Even equation 12 tells us that, although the conditional variance of the residuals will vary with time, it will stabilize, unless it is fed with exogenous shocks. The intrinsic instability of the model is not captured by any of the components of the EGARCH estimates.

5. The BDS Statistic

Although the erratic behaviour between a chaotic and a random motion may seem indistinguishable, there are important differences that allowed Brock et al. (1987)[16] to propose a statistical procedure to test departures from independently and identically distributed (i.i.d.) observations.

Consider T observations of a time series ($x_1, x_2, \ldots, x_T$) after removing all non-stationary components. Define the m-histories of x_t process as the vectors ($x_1, \ldots, x_m$), ($x_2, \ldots, x_{m+1}$), ..., ($x_{T-m+1}, \ldots, x_T$). The correlation integral is the fraction of the distinct pairs of m-histories lying within a distance ε in the sup norm[17]:

$$C_{\varepsilon,m,T} = \frac{\sum_{i=1}^{T}\sum_{j=1}^{T} H\left(\varepsilon - \sup norm\left(\mathbf{x}_i, \mathbf{x}_j\right)\right)}{(T-m+1)(T-m)} \quad (13)$$

(with $i \neq j$)

where $\mathbf{x}_i = \left(x_i, \ldots x_{i+m-1}\right)$, and $H(x) = \begin{cases} 0 & if\, x \leq 0 \\ 1 & if\, x > 0 \end{cases}$

Under some assumptions $C_{\varepsilon,m,T}$ converges to $C_{\varepsilon,m}$ (see Brock et al. (1991) for details). The true correlation dimension is $\frac{d\ln\left(C_{\varepsilon,m}\right)}{d\ln(\varepsilon)}$. It is possible to show that the correlation dimension never exceeds the Haussdorff dimension. If as *m* increases $\frac{d\ln\left(C_{\varepsilon,m,T}\right)}{d\ln(\varepsilon)}$ also increases, then the system is stochastic. If, however, it tends to a constant then the data is consistent with chaotic behavior.

Brock et al. (1987)[18] employed the correlation dimension to obtain a statistical test of non-linearity: they proved that under the null (x_t i.i.d.) $\ln\left(C_{\varepsilon,m}\right) = m\ln\left(C_{\varepsilon,1}\right)$, which is the basis for the BDS statistic:

16 Brock, W. Dechert, W. and Sheinkman, J. (1987), "A Test for Independence Based on the Correlation Dimension", University of Wisconsin, Madison, University of Houston, and University of Chicago, cit. in Brock et al. (1991).
17 Brock (1986) showed that the correlation dimension was independent of the choice of the norm, so it is not restrictive to consider the sup norm.
18 See footnote 16.

$$BDS = \sqrt{T}\frac{C_{\varepsilon,m,T} - \left(C_{\varepsilon,1,T}\right)^m}{\sigma_{\varepsilon,m,T}} \quad (14)$$

where $\sigma_{\varepsilon,m,T}$ is the standard deviation consistently estimated[19]. Under the null the BDS has a limiting standard normal distribution. The asymptotic distribution behaves reasonably well, if the sample size is not less than 500, but it is poor for smaller sample dimensions.

To implement the BDS test, Monte Carlo Simulations of Brock et al. (1991) suggest that ε should vary between 0.5 and 2 standard deviations of the data, and m between 2 and 5.

5.1. The BDS Statistic applied to our model

Since an ARCH model and its extensions typically assume i.i.d. standard residuals, Bollerslev et al. (1994) suggest the use of the BDS test as a specification test applied to the standardized residuals of a model. We have already seen that the Kiefer-Salmon test applied to the standardized residuals of our EGARCH(2, 4) did not reject the normality of those residuals. I now apply the BDS test to the same residuals. Two difficulties need to be faced with: the small dimension of the sample, and the asymptotic distribution of the test, which is strongly affected by the fitting of the EGARCH model, and has not been derived yet. To overcome both problems, I follow a procedure suggested by Brock et al. (1991), also applied by Louçã (1997): after estimating the BDS statistic I shuffle randomly the time series sample and then re-estimate the statistic. This procedure is repeated 100 times. If the process is purely random the dimension of the process will be unchanged and so will the estimated statistic. If the process is purely deterministic, then shuffling will destroy the correlation structure of the process. In table 3 we can see the results achieved[20]. The null hypothesis is, correctly, rejected.

Table 3: BDS Test to the EGARCH standard residuals

m	2	3	4	5
$\varepsilon = 0.5\sigma$	13.85 (0.00)	18.01 (0.00)	25.44 (0.00)	41.47 (0.00)
$\varepsilon = \sigma$	7.16 (0.00)	7.14 (0.00)	7.37 (0.00)	7.8 (0.00)

5.2. Some Problems

The above results suggest that it is easy to determine whether a time series follows a chaotic process or not. This is not correct. First, there is no practical distinction between a high dimensional chaotic model and a pure stochastic model, so this test is only appropriate to detect low dimensional chaos. Second, the rejection of the null hypothesis does not tell us anything about the alternative. For example, the data generator process may be a stochastic non-linear model and not a chaotic deterministic model. So the BDS statistic can also be used as a specification test for any non-linear model. The rejection of the null only tells us that there are some hidden non-linearities not captured by the original model.

Another interesting problem, particularly when we are analyzing macroeconomic time series, is the problem with aggregate data. Goodwin (1991) defended the use of large multidimensional disaggregated systems even though, unfortunately, he had always worked with aggregated models.

19 See Brock et al. (1991) for details on how to consistently estimate the standard deviation.

20 In parenthesis we have the proportion of the statistic values (obtained after reshuffling) that are higher (in absolute value) than the statistic value of the original series.

Table 4: BDS Test to Chaotic Time Series

m	2		3		4		5	
	$\varepsilon = 0.5\sigma$	$\varepsilon = \sigma$	$\varepsilon = 0.5\sigma$	$\varepsilon = \sigma$	$\varepsilon = 0.5\sigma$	$\varepsilon = \sigma$	$\varepsilon = 0.5\sigma$	$\varepsilon = \sigma$
A_t	709	287	939	273	1239	262	1690	265
B_t	691	286	915	265	1191	252	1613	244
C_t	733	285	969	270	1265	259	1711	252
D_t	692	287	925	269	1218	260	1657	254
E_t	690	288	912	271	1190	259	1598	252
F_t	0.07	−1.08	1.02	−1.36	2.18	−1.25	1.57	−1.18

To illustrate this problem we can see in table 4 the BDS test applied to five different chaotic series[21] and to their average ($F_t = (A_t + B_t + C_t + D_t + E_t)/5$). Since I constructed a sample of 2000 observations the normal distribution may be used to find the critical values. While for any of the series obtained from a logistic chaotic equation there is overwhelming evidence of non-linearities, for the average of five chaotic series that evidence has almost completely disappeared: it is impossible to reject the null hypothesis at a 5% significance level except for $(\varepsilon, m) = (0.5\sigma, 4)$. These results are not surprising: F_t is, obviously, chaotic, but being an average of five different chaotic series, it has a higher dimension than the primitive series, being harder to distinguish from a purely random motion.

6. Conclusion

We saw that the traditional econometric techniques have some flaws when dealing with deterministic chaos. Using the traditional econometric approach one will tend to accept that the source of the erratic movements is exogenous and that the system is dynamically stable, even though the model is known to be inherently unstable.

A different problem is that specific econometric techniques, designed to deal with the possibility of deterministic chaos, are not as powerful as one might wish: aggregation can hide evidence of non-linearities (a problem that can arise in many macroeconomic time-series), and the alternative to the null hypothesis is not well defined.

Some of these problems could be overcome if longer (with some thousands of observations) macroeconomic time series were available but that, unfortunately, is unavailable. Probably these are the reasons why there is so much more evidence of chaos in financial time series[22] than in macroeconomic data: in financial time series there are huge data sets at an extremely disaggregated level.

In my opinion there is no fundamental reason to think that the financial system is deeply different from the general economic system, and so I see no reason to disregard *a priori* the study of the consequences of chaos in macroeconomic models.

Interestingly the economic science is not in harmony with other mathematical sciences. In Physics and Chemistry (from which much of our methodology has been imported) and in Biology, complex systems are the rule and not the exception. If we accept that economic life is nonlinear and that the oscillators are not independent but rather they interact between them, then chaos may come out. More precisely, in a continuous system we need at least three oscillators linked in order to produce chaos (e.g. Rossler band in Goodwin's model). Dechert et al. (1999) report the

21 The series were generated according to the formula: $X_t = 4X_{t-1}(1 - X_{t-1})$. The only difference between A_t, B_t, C_t, D_t, and Et were the initial values, which are strictly contained between zero and one.
22 E.g., see Scheinkman and LeBaron (1989), and also Serletis and Gogas (1997).

result of a Monte Carlo study on the probability of chaos in large dynamical systems. They used neural networks as proxies for the equations that describe the dynamics of the system. Their results were quite impressive: “as the dimension of the system and the complexity of the network increase, the probability of chaotic dynamics increases to 100%”. It does not seem unrealistic to believe this is the case of the economic system with heterogeneous agents, imperfect markets, monopoly power, commercial and political relations between countries with different economic and political systems.

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