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The choice of a growth path under a linear quadratic approximation

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resumo

Existe uma corrente recente de literatura que aponta para a relevância do método de aproximação linear quadrática da função objectivo na vizinhança do ponto de equilíbrio, nomeadamente quando se assume um cenário estocástico. Por exemplo, aplicações no campo da política monetária recorrem a esta técnica. Neste artigo verifica-se que, para um problema de controle óptimo específico, sob uma estrutura puramente determinística, uma aproximação de segunda ordem da função objectivo pode conduzir a resultados inexactos, particularmente quando se consideram variáveis exógenas como argumentos da função objectivo. Estes resultados estão relacionados com condições de estabilidade, as quais, no presente caso, podem ser escritas como restrições ao valor da taxa de desconto de resultados futuros. O modelo proposto é designado por modelo de 'controle óptimo de crescimento'; para este modelo, são calculadas condições gerais de estabilidade e discutida uma aplicação da estrutura teórica a um problema de fertilidade – capital humano.

résumé / abstract

Un courant récent de la littérature indique l'importance de la méthode de rapprochement linéaire quadratique de la fonction objectif au voisinage du point d'équilibre, notamment dans un scénario stochastique. Par exemple, des applications dans le domaine de la politique monétaire ont recours à cette technique.
Dans cet article, nous constatons que, pour un problème de contrôle optimal spécifique, sous une structure sans incertitude aucune, un rapprochement de second ordre de la fonction objectif pourrait conduire à des résultats inexacts, notamment quand nous considérons des variables exogènes comme arguments de la fonction objectif. Ces résultats sont dus à des conditions de stabilité qui, dans le cas présent, peuvent correspondre à des contraintes sur la valeur du taux de remise de futurs résultats. Le modèle considéré est désigné par modèle de "contrôle optimal de croissance" et, pour ce modèle, on a calculé des conditions générales de stabilité ainsi que discuté une application de la structure théorique à un problème de fertilité et de capital humain.

There is a recent strand of literature which suggests that second order approximations of linear quadratic objective functions in the steady state vicinity, namely when assuming stochastic scenarios, lead to very interesting and useful results. For example, applications in monetary policy resort to such technique.
In this paper we find that, for a specific optimal control problem under a purely deterministic setup, a second-order approximation of the objective function may lead to inaccurate results, particularly when one considers exogenous variables as arguments of the objective function. These results are related to the stability conditions, which in the present case can be written as constraints to a discount rate associated with future outcomes. We designate the proposed model as an 'optimal growth control' model, from which we compute general conditions about stability and analyse the application of such a framework to a fertility – human capital problem.

JEL Classification: C61; C62; J13; O41

1. Introduction[1]

Recent work in economic policy has focused on second-order (linear quadratic) approximations to the objective function in order to evaluate the dynamics of the models under discussion. The use of second-order approximations emerges as a way to overcome some limitations that traditional methods of analysis have shown. In the words of Schmitt-Grohé and Uribe (2002),

"(...) if the support of the shocks driving aggregate fluctuations is small and an interior stationary solution exists, first-order approximations provide adequate answers to questions such as local existence and determinacy of equilibrium and the size of the second moments of the variables describing the economy. However, first-order approximation techniques are not well suited to handle questions such as welfare comparisons across alternative stochastic environments." (2002: 1).

Linear quadratic methods, as developed by Judd (1998), Sims (2000) and Collard and Juillard (2001), have a widespread use today, with particular emphasis on the analysis of monetary policy, as discussed in Woodford (2001), Sutherland (2002) and Benigno and Woodford (2004).

The use of a second-order approximation to the objective function is not made in an ad hoc way, that is, it obeys to some criteria. First of all, it implies that the objective function and the production possibilities are presented under a generic form rather than considering explicit functional forms; this has the obvious advantage of producing general results, which are robust and not tied down to the specific properties of the assumed functional form. In this respect, Woodford (2003) presents clear and convincing arguments:

"A common approach in the quantitative equilibrium business-cycle literature, of course, is to assume special functional forms for preferences and technology that allow the higher derivatives of these functions to be inferred from the same small number of parameters as determine the lower-order derivatives, which may then be inferred from first and second moments of the time series alone. This approach often obscures the relation between the properties of the time series and the model parameters that are identified by them, and allows 'identifications' that are in fact quite sensitive to the arbitrary functional form assumption. I prefer instead to assume functional forms that are as general as possible and then to emphasize that only a finite number of derivatives of these functions matter for the calculations." (2003: 384, ft. 5).

The statement by Woodford also sheds light over the reason why higher than second-order approximation should not be used. The reason has to do with a parsimony criterion. A Taylor--series expansion of an order higher than two would imply the introduction of a great number of additional parameters, which would make the analysis harder, namely when assigning values in order to proceed with numerical simulations. Generally, a Taylor-series expansion of order two entails the sufficient degree of precision and accuracy to characterize the dynamic behavior of economic variables and to evaluate policy decisions.

The application of a linear quadratic method in the evaluation of economic dynamic models implies the use of a Taylor-series expansion. As Díaz-Giménez (1998) emphasizes, the method requires the consideration of a steady state point, because it is around this point that the objective function can be expanded. Therefore, the linear quadratic approach should be applied in circumstances where: (i) the deterministic version of the model converges to a stable steady state; (ii) the local dynamics around the steady state are well approximated by a linear law of motion.

Woodford (2003) also presents the following conditions required for the validity of the second--order approximation method: (i) the objective function must be a concave function, relatively to their endogenous variables; (ii) the objective function must be at least twice differentiable.

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In this paper, the linear quadratic approach is applied to a simple optimal control problem in continuous time and infinite horizon framework. The basic aim is to make an explicit point about this approach; comparing, for our simple model, the linear quadratic technique results to a standard linear analysis, one will be able to understand that the first has a simplicity advantage, in the sense that it allows for a straightforward computation of the Jacobian matrix, but it has also a disadvantage that can be a relevant problem: when considering exogenous variables in the objective function, it neglects some terms that can influence stability conditions.

In fact, under the continuous time-infinite horizon framework as considered here, we will be able to see that the *difference* between a second-order approach and a linear treatment of the assumed specific model is relevant only if one considers exogenous variables in the objective function. Therefore, this purely deterministic continuous time framework is not the ideal scenario for applying a linear quadratic approximation. To develop our arguments, the analysis will proceed in two stages:

– Firstly, exogenous variables are ignored in the objective function, and we will show that the second-order method yields exactly the same results as a trivial linear analysis;

– Secondly, we introduce an exogenous variable in the objective function, and as a result the linear quadratic method leads to inaccuracy which is not the case in the linear approach.

One may conclude that there are cases in which the linear quadratic method should not be used in order to obtain results which are expected to be accurate and robust. The only reason that may justify the application of such a procedure is that it simplifies the computation, nevertheless it may lose relevant information which is necessary in order to evaluate stability in a rigorous way.

The optimal control problem considered here in order to discuss the linear quadratic approximation will be designated as the 'optimal growth control (OGC) model'. In the OGC class we will include models that obey the following characteristics:

(i) There is a variable that grows in time at a constant rate;

(ii) There is a second variable that grows at a rate that can be controlled by the representative agent;

(iii) The representative agent takes an infinite horizon and maximizes an objective function, which has the following arguments: the controllable growth rate, and the ratio of the variable that grows at a constant rate with respect to the variable which controls the correspondent growth rate.

The mathematical properties of the OGC model are derived and discussed. Firstly, the model has a unique steady state. The steady state solution implies the existence of a common growth rate for both endogenous variables (the state and the control variable) and, necessarily, there is a constant value for the ratio of the two variables. Having defined the steady state point, we study the model's dynamics through the consideration of a second-order approximation of the objective function around the steady state point. The use of a second-order Taylor expansion to approximate the objective function will be compared with a standard linear approach and, as stated earlier, important differences will be found only when one introduces exogenous variables into the objective function.

Relatively to the dynamics of the OGC model, we found that the correspondent dynamic system is unstable unless the discount rate relating future outcomes is bounded. If the saddle-path stability condition holds (that is, if the referred bound is satisfied), then we are able to discuss the impact of exogenous perturbations. In order to illustrate the model's results, an economic example that relies on the structure of the OGC model will be presented. Specifically, a fertility-human capital model is developed.

The remainder of the paper is organized as follows. Section 2 presents the OGC model and approximates the objective function in the steady state vicinity. Section 3 solves the model in order to find a condition under which saddle-path stability holds; this condition takes the form of a bound on the discount rate. The results are compared with a linear treatment of the model, and no significant changes are found – a similar pair of differential equations is determined and, therefore, the results are not changed. In section 4, the short-run and the long-run effects of changes in parameter values are evaluated. Section 5 introduces an exogenous variable into the objective function; and in this case, the linear quadratic approach leads to an inaccurate set of results. In particular, a term in the discount rate bound to stability will be missing. Section 6 develops a simple example of the OGC model, that relates the options of households regarding the number of children, on one hand, and providing their children with good education standards on the other. Finally, section 7 concludes.

2. The OGC Model

Let parameter a be the growth rate of a variable $A(t) \geq 0$, $A(0) = A_0$ given. Let variable $\beta(t) \in \boldsymbol{R}$ be the growth rate of variable $B(t) \geq 0$, $B(0) = B_0$ given. The representative agent intends to maximize intertemporally the value of a function v: $\boldsymbol{R}^2 \rightarrow \boldsymbol{R}$, where the arguments are variables $\beta(t)$ and $\alpha(t) \equiv A(t)/B(t)$. Assuming a discount rate $\rho > 0$, the problem is

$$(1) \qquad \underset{\beta(t)}{Max} \int_0^{+\infty} v[\alpha(t), \beta(t)].e^{-\rho.t}.dt \text{ subject to } \dot{A}(t) = a.A(t),\ \dot{B}(t) = \beta(t).B(t)$$

with v as a concave and smooth function (continuously many times differentiable). This is a very simple model, but it has important properties and allows us to reach meaningful results.

To solve the model, we define $\alpha(t)$ in such a way that it transforms the two state constraints in only one constraint

$$(2) \qquad \dot{\alpha}(t) = [a - \beta(t)].\alpha(t)$$

State constraint (2) reflects a first important point in the model: in the steady state both variables, $A(t)$ and $B(t)$, grow at rate a, and as corollary it implies that $\alpha(t)$ has to be a constant value in the long run equilibrium. Thus, the steady state will be a pair of values $(\bar{\alpha}, \bar{\beta}) = (k, a)$, where k is some positive constant that can be determined by solving the model.

Given that there is a unique steady state result, we may approximate function v around this steady state; a second-order approximation is considered and higher-order terms are neglected. Hence, we take

$$(3) \qquad v[\alpha(t), \beta(t)] = \theta_0 + \theta_1.\alpha(t) + \theta_2.\beta(t) + \\ + \theta_3.\alpha(t).\beta(t) + \theta_4.\alpha(t)^2 + \theta_5.\beta(t)^2 + O(\|\alpha(t), \beta(t)\|^3)$$

with $\theta_i \in \boldsymbol{R}$, i = 0,…, 5. The term $O(\|\alpha(t), \beta(t)\|^3)$ translates the higher-order terms that are not essential for the subsequent analysis. To produce a meaningful analysis, we also define the following parameters: $\theta'_1 = \theta_1 + \theta_3.a + 2.\theta_4 k$ and $\theta'_2 = \theta_2 + \theta_3.k + 2.\theta_5.a$.

3. A Condition for Saddle-Path Stability

To solve the OGC model, one considers a current-value Hamiltonian function, with $p(t)$ a co-state variable,

$$(4) \qquad H[\alpha(t), \beta(t), p(t)] \equiv v[\alpha(t), \beta(t)] + p(t).[a - \beta(t)].\alpha(t)$$

The first-order optimality conditions are,

(5) $\theta_2 + \theta_3.\alpha(t) + 2.\theta_5.\beta(t) = p(t).\alpha(t)$

(6) $\dot{p}(t) = [\rho + \beta(t) - a].p(t) - \theta_1 - 2.\theta_4.\alpha(t) - \theta_3.\beta(t)$

(7) $\lim_{t \to +\infty} \alpha(t).e^{-\rho.t}.p(t) = 0$

Condition (7) is a transversality condition. This is a limit condition that guarantees efficiency at the end of the considered horizon (it says that the quantity of the state variable left at the end, discounted at a rate ρ, is zero, or, otherwise, the correspondent shadow-price is zero). We consider an infinite horizon, and therefore this condition holds asymptotically. Pitchford (1977) and Benveniste and Scheinkman (1982) have shown that the transversality condition is a necessary condition for optimization whenever one considers a continuous time model with time discounting, as it is the case in this paper.

Optimality conditions allow for the determination of a differential equation describing the time movement of $\beta(t)$. Differentiating (5) with respect to time, it is possible to establish that

(8) $\dot{\beta}(t) = \frac{1}{2.\theta_5}.\{\alpha(t).\dot{p}(t) + [p(t) - \theta_3]\dot{\alpha}(t)\}$

Replacing (2) and (6) in (8) and taking in consideration that the value of $p(t)$ can be withdrawn from (5), the following equation is the intended dynamic expression,

(9) $\dot{\beta}(t) = \frac{1}{2.\theta_5}.\{\theta_2.\rho + [(\rho - a).\theta_3 - \theta_1].\alpha(t) - 2.\alpha(t)^2 + 2.\theta_5.\rho.\beta(t)\}$

To solve the dynamics of the OGC model we consider equations (1) and (9), implying that this system of two equations involves our two endogenous variables, $\alpha(t)$ and $\beta(t)$.

Note that in the presence of (9), one can determine the value of k. Taking $\dot{\beta}(t) = 0$, one computes $k = \frac{\theta_2'}{\theta_1'}.\rho$. As far as k is concerned, an important feature is that for this variable to be positive, θ_1' and θ_2' have to have the same sign. Such condition has important implications regarding the slope of an eventual saddle-path trajectory, as one will verify bellow.

To proceed with the dynamic analysis, we linearize the model around the steady state result. We get,

(10) $\begin{bmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \end{bmatrix} = \begin{bmatrix} 0 & -k \\ \omega & \rho \end{bmatrix} . \begin{bmatrix} \alpha(t) - \overline{\alpha} \\ \beta(t) - \overline{\beta} \end{bmatrix}$

with $\omega = \frac{1}{2.\theta_5}.\left[\theta_3.\rho - 2.\frac{\theta_1'}{\theta_2'}\ \theta_4.\rho - \theta_1'\right].$

The qualitative behavior of the system depends on the signs of the eigenvalues of the matrix in (10). These are the solution for the characteristic equation $\lambda^2 - \rho.\lambda + k.\omega = 0$. Noticing that

$Tr(J) = \rho$ and $Det(J) = k.\omega$, with J the 2x2 matrix in (10), we immediately conclude that the system is unstable unless $\omega < 0$. If this condition is satisfied we have a saddle-path equilibrium and a stable arm through which endogenous variables can converge to the equilibrium point is computable. Because the steady state gives the optimal growth rate and the optimal share of the model's variables, it is our main interest to determine the condition under which the saddle-path stability holds. We will present this condition as a bound to the discount rate – depending on the sign of θ_5, which can be an upper bound or a lower bound. The following is the condition for saddle-path stability

(11) $$\left(\theta_5 > 0 \text{ and } \rho < \frac{(\theta_1')^2}{\theta_1'.\theta_3 - 2.\theta_2'\theta_4}\right) \text{ or } \left(\theta_5 > 0 \text{ and } \rho > \frac{(\theta_1')^2}{\theta_1'.\theta_3 - 2.\theta_2'\theta_4}\right)$$

According to (11), if θ_5 is a positive value, saddle-path stability implies a low discount rate for future achievements; and if the referred constant is negative we have to consider a high discount rate as a means to guarantee stability.

Assuming that (11) holds, one finds the stable trajectory through the computation of the eigenvector that is associated to the negative eigenvalue of J, which is given by $P(\lambda_1) = [1 - \lambda_1 / k']$. The second element of the vector is the slope of the stable trajectory, and therefore the stable trajectory may be presented as follows

(12) $$\beta(t) - \bar{\beta} = -\frac{\lambda_1}{k}.[\alpha(t) - \bar{\alpha}]$$

Taken into account that λ_1 is the negative eigenvalue of J, and thus variables $\alpha(t)$ and $\beta(t)$ will converge along the stable arm to the steady state value in a same qualitative way – both variables will have their values rising/falling – as the system approaches the steady state.

Instead of resorting to the linear quadratic approach, we may solve the OGC model without assuming a second-order expansion over the objective function. Given the Hamiltonian function (4), the necessary optimality conditions can be presented in the generic form

(13) $$v_\beta = p(t).\alpha(t)$$

(14) $$\dot{p}(t) = [\rho + \beta(t) - a].p(t) - v_\alpha$$

with v_a and v_β the first-order derivatives of the objective function. The transversality condition (7) continues to hold.

Differentiation of (13) with respect to time implies the following condition:

(15) $$v_{\beta\beta}\dot{\beta}(t) + v_{\alpha\beta}\dot{\alpha}(t) = \dot{p}(t).a(t) + p(t).\dot{\alpha}(t)$$

with $v_{\alpha\alpha}$ and $v_{\alpha\beta}$ second-order derivatives of the objective function. Rearranging (15) and replacing the derivative of $p(t)$ by the correspondent expression in (14) and $\dot{\alpha}(t)$ by the expression in (2) one obtains the differential equation

(16) $$\dot{\beta}(t) = \frac{1}{v_{\beta\beta}}[v_\beta.\rho - v_\alpha.\alpha(t)] - \frac{v_{\alpha\beta}}{v_{\beta\beta}}.[a - \beta(t)].\alpha(t)$$

When evaluated in the steady state vicinity, (16) does not differ from (9), if one takes the following correspondences: $\theta_1' = \overline{v}_{\alpha}$, $\theta_2' = \overline{v}_{\beta}$, $\theta_3 = \overline{v}_{\alpha\beta} = \overline{v}_{\beta\alpha}$, $\theta_4 = \frac{1}{2}.\overline{v}_{\alpha\alpha}$, and $\theta_5 = \frac{1}{2}.\overline{v}_{\beta\beta}$. Also, the previous identities imply that a same k steady state value for $\alpha(t)$ is found and equal derivatives in the linearized system can be determined, that is, $\left.\frac{\partial\dot{\beta}(t)}{\partial\alpha(t)}\right|_{(\overline{\alpha},\overline{\beta})}$ and $\left.\frac{\partial\dot{\beta}(t)}{\partial\beta(t)}\right|_{(\overline{\alpha},\overline{\beta})}$ are the same as in (10). Subsequently, the linear approach must lead to a same steady-state condition as in (11) and to a stable trajectory equal to (12).

For now, our main conclusion is that, in the absence of exogenous variables in the objective function, the dynamic analysis in the steady state vicinity does not differ, qualitatively, if we consider a linear quadratic approximation or a linear approximation. Nevertheless, the first method is more general in the sense that the various parameters do not have to represent necessarily first and second order derivatives of the objective function. Thus, a second order approximation of the kind introduced in this analysis is appropriate when one does not know the functional form of the objective function and, consequently, we are not able to compute first and second order derivatives.

4. Exogenous Disturbances

Assume that condition (11) is satisfied. In such circumstance, one is able to evaluate the impact of a change in the discount rate or a change in the growth rate a over the OGC model's dynamics. Rewriting the model in order to include disturbances over ρ and a,

$$(10)\quad \begin{bmatrix}\dot{\alpha}(t)\\ \dot{\beta}(t)\end{bmatrix} = J.\begin{bmatrix}\alpha(t)-\overline{\alpha}\\ \beta(t)-\overline{\beta}\end{bmatrix} + D.\begin{bmatrix}d\rho\\ da\end{bmatrix}, \text{ with } D = \begin{bmatrix}0 & k\\ -\omega.\frac{k}{\rho} & -\rho\end{bmatrix}$$

In (17), the elements in the 2x2 matrix D are the derivatives of each of the dynamic equations, (2) and (9), relatively to each of the parameters ρ and a; and these derivatives are evaluated in the steady state. From (17), we can measure the effect of a perturbation in ρ and a over the steady state values of $\alpha(t)$ and $\beta(t)$. Such disturbances will imply a short-run and a long-run effect. The long-term effect is a jump of the control variable from the initial saddle trajectory to a new trajectory, through which variables adjust to the long-run steady state solution that is formed after the disturbance takes place.

Long-run multipliers are determined using the formula $\begin{bmatrix}\Delta\overline{\alpha}(\infty)\\ \Delta\overline{\beta}(\infty)\end{bmatrix} = -J^{-1}.D.\begin{bmatrix}d\rho\\ da\end{bmatrix}$. Computation leads to

$$(18)\quad \begin{bmatrix}\Delta\overline{\alpha}(\infty)\\ \Delta\overline{\beta}(\infty)\end{bmatrix} = \begin{bmatrix}k/\rho & 0\\ 0 & 1\end{bmatrix}.\begin{bmatrix}d\rho\\ da\end{bmatrix}$$

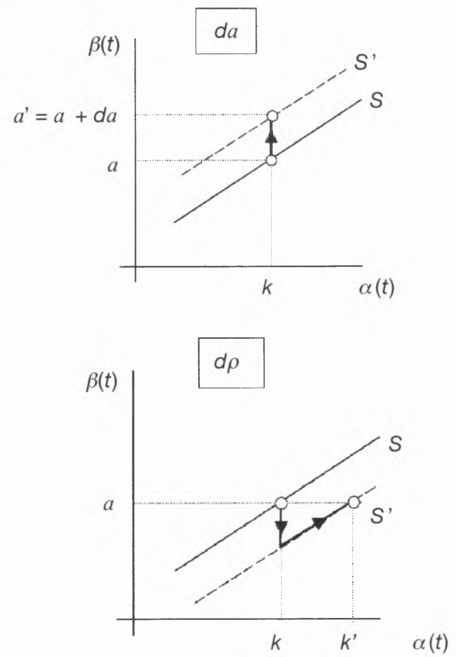
According to (18), a positive change in the discount rate produces a positive change in the equilibrium value of $\underline{\alpha}(t)$. Relatively to the disturbance in a, this just produces a change of exactly the same amount in $\overline{\beta}$.

Short-run multipliers are computable only for the control variable, and correspond to

$$(19)\quad \begin{bmatrix}\Delta\overline{\alpha}(0)\\ \Delta\overline{\beta}(0)\end{bmatrix} = \begin{bmatrix}0 & 0\\ \lambda_1/\rho & 1\end{bmatrix}.\begin{bmatrix}d\rho\\ da\end{bmatrix}$$

The short-run effect of a disturbance in a implies exactly the same result as the long-run effect; thus, in the case of a change in the value of growth rate a, the steady state point is immediately transferred to a new point, without any change in the value of k. Relatively to the short-run effect of a change in the discount rate, this will consist in a change of negative sign in the steady state value of $\beta(t)$ to a new saddle-path; once the new saddle-path is reached, the variable will converge exactly to the same value as in the original steady-state. Figure 1 represents the possible impact of positive changes in the values of a and ρ.

Figure 1 – Steady state perturbations implied by changes in the values of a and ρ



In figure 1, the top panel describes the effect of a positive change in growth rate a, as a result of the change in this parameter value, and shows that the saddle-path shifts from S to S', but no change is observed in the steady state value of $\alpha(t)$ – the new steady state is immediately accomplished through a jump in the value of β. The second panel shows the impact of a positive change in the value of the discount rate, implying an initial jump in variable $\beta(t)$ from saddle-path S to saddle-path S', and thereafter an adjustment process towards the new equilibrium value of $\alpha(t)$.

5. The Introduction of an Exogenous Variable

Consider now an exogenous variable $\sigma(t)$, which we assume that grows at some unspecified rate, $\dot{\sigma}(t) = s(\sigma(t))$, $\sigma(0) = \sigma_0$ given. This variable is included in the objective function, $v[\alpha(t), \beta(t), \sigma(t)]$. We continue to consider that function v is smooth and concave, and that symmetry properties hold, that is, $v_{\alpha\beta} = v_{\beta\alpha}$, $v_{\alpha\sigma} = v_{\sigma\alpha}$ and $v_{\beta\sigma} = v_{\sigma\beta}$. To solve the optimization problem of the previous sections with this new variable, one proceeds in exactly the same way: the optimal control problem can be solved for the original objective function and a linearization process is considered after finding optimality conditions. Alternatively, it is possible to proceed in a first step with a second-order approximation of the objective function.

Let us start by finding optimality conditions for the non approximated objective function. Given a Hamiltonian function similar to (4), one finds once again the optimal relations (13) and (14). The main difference relatively to the previous version of the model is that time differentiation of (13) yields a slightly different version of (15), which includes a term relating to the exogenous variable and, thus, differential equation (16) has to be replaced by

$$(20) \quad \dot{\beta}(t) = \frac{1}{v_{\beta\beta}}[v_{\beta}.\rho - v_{\alpha}.\alpha(t) - v_{\beta\sigma}.s(\sigma(t))] - \frac{v_{\alpha\beta}}{v_{\beta\beta}}.[a - \beta(t)].\alpha(t)$$

Evaluating (20) in the steady state neighbourhood, one observes the following steady state value for $\alpha(t)$:

$$(21) \quad \overline{\alpha}(t) = \frac{\overline{v}_{\beta}}{\overline{v}_{\alpha}}.\rho - \frac{\overline{v}_{\beta\sigma}}{\overline{v}_{\alpha}}.s(\overline{\sigma})$$

Note that the steady state result (21) is equal to the steady state value found in previous sections only if one assumes that the exogenous variable, $\sigma(t)$, does not grow in a long-run perspective. Since initially we have imposed the constraint $\alpha(t) \geq 0$, now this requires that the following condition must be satisfied

$$(22) \quad \rho > \frac{\overline{v}_{\beta\sigma}}{\overline{v}_{\beta}}.s(\overline{\sigma})$$

The Jacobian matrix also suffers some changes, except in the special case of no growth of the exogenous variable. Now, the Jacobian matrix can be written as

$$(23) \quad J' = \begin{bmatrix} 0 & -[\frac{\overline{v}_{\beta}}{\overline{v}_{\alpha}}.\rho - \frac{\overline{v}_{\beta\sigma}}{\overline{v}_{\alpha}}.s(\overline{\sigma})] \\ \omega' & \rho - \frac{v_{\beta\sigma\beta}}{v_{\beta\beta}}.s(\overline{\sigma}) \end{bmatrix}, \text{ with}$$

$$\omega' = \frac{1}{v_{\beta\beta}}.\left[\left(\overline{v}_{\alpha\beta} - \frac{\overline{v}_{\beta}}{\overline{v}_{\alpha}}.\overline{v}_{\alpha\alpha}\right).\rho - \overline{v}_{\alpha} + \left(\frac{\overline{v}_{\beta\sigma}}{\overline{v}_{\alpha}}.\overline{v}_{\alpha\alpha} - \overline{v}_{\beta\sigma\alpha}\right).s(\overline{\sigma})\right]$$

with $v_{\beta\sigma\beta}$ and $v_{\beta\sigma\alpha}$ third-order derivatives of the objective function. Some important differences are identifiable with respect to the original setup. Firstly, now we will have a more sophisticated saddle-path equilibrium condition. A second novelty is that full stability (in the steady state vicinity) eventually exists. If the trace of *J'* is a negative value, what is now a possibility, and the determinant is positive, then the steady state is reached, independently of the position of the system's initial state (being this initial state a point in the steady state vicinity). Thus, conditions for full stability are, besides (22):

$\rho > \frac{\overline{v}_{\beta\sigma\beta}}{\overline{v}_{\beta\beta}}.s(\overline{\sigma})$ and one of the following:

$$(24) \quad i)\ \overline{v}_{\beta\beta} > 0;\ \rho > \frac{\overline{v}^2_{\alpha} + (\overline{v}_{\alpha}.\overline{v}_{\beta\sigma\alpha} - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta\sigma}).s(\overline{\sigma})}{\overline{v}_{\alpha\beta}.\overline{v}_{\alpha} - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta}}$$

$ii)\ \overline{v}_{\beta\beta} < 0;\ \rho > \dfrac{\overline{v}^2{}_\alpha + (\overline{v}_\alpha.\overline{v}_{\beta\sigma\alpha} - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta\sigma}).s(\overline{\sigma})}{\overline{v}_{\alpha\beta}.\overline{v}_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_\beta}$

According to (24) there are some possibilities for the value of the discount rate under which the model's dynamics correspond to a stability result. Saddle-path stability implies a negative determinant, and therefore one of the following set of conditions must be satisfied:

$i)\ \overline{v}_{\beta\beta} > 0;\ \rho > \dfrac{\overline{v}^2{}_\alpha + (\overline{v}_\alpha.\overline{v}_{\beta\sigma\alpha} - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta\sigma}).s(\overline{\sigma})}{\overline{v}_{\alpha\beta}.\overline{v}_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_\beta}$

(25)

$ii)\ \overline{v}_{\beta\beta} < 0;\ \rho > \dfrac{\overline{v}^2{}_\alpha + (\overline{v}_\alpha.\overline{v}_{\beta\sigma\alpha} - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta\sigma}).s(\overline{\sigma})}{\overline{v}_{\alpha\beta}.\overline{v}_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_\beta}$

Finally, instability implies a positive trace and a positive determinant of J'. Stability conditions continue to demand a bound over the discount rate, but the introduction of the exogenous variable introduces a variety of new possibilities, including the case in which full stability can be achievable.

Our main point in this paper is precisely the idea that the linear quadratic approach departs from the previous technique when assuming the presence of the exogenous variable in the objective function. As we shall see bellow, the second-order expansion means a computation procedure that is simpler but where some terms just disappear from the analysis. Therefore, our conclusion is that for a simple deterministic continuous time intertemporal model, the linear quadratic procedure does not lead to fully accurate results.

Let us develop the linear quadratic approach. The second-order expansion of the objective function allows us to write

$$\begin{aligned} v[\alpha(t), \beta(t), \sigma(t)] = {} & \eta_0 + \eta_1.\alpha(t) + \eta_2.\beta(t) + \eta_3.\sigma(t) + \\ & + \eta_4.\alpha(t).\beta(t) + \eta_5.\alpha(t).\sigma(t) + \eta_6.\beta(t).\sigma(t) + \\ & + \eta_7.\alpha(t)^2 + \eta_8.\beta(t)^2 + \eta_9.\sigma(t)^2 + O(\|\alpha(t), \beta(t), \sigma(t)\|^3) \end{aligned} \tag{26}$$

with $\eta_i \in \boldsymbol{R}$, $i = 0, \ldots, 9$. We also consider the following values,

$$\eta_1' = \eta_1 + 2.\eta_7.\overline{\alpha} + \eta_4.a + \eta_5.\overline{\sigma},$$

$$\eta_2' = \eta_2 + \eta_4.\overline{\alpha} + 2.\eta_8.a + \eta_6.\overline{\sigma},$$

$$\eta_3' = \eta_3 + \eta_5.\overline{\alpha} + \eta_6.a + 2.\eta_9.\overline{\sigma}.$$

Computing optimality conditions, the following equation is derived

$$\begin{aligned} \dot{\beta}(t) = \frac{1}{2.\eta_8}.\{ & \rho.\eta_2 + [(\rho - a).\eta_4 - \eta_1].\alpha(t) + 2.\eta_8.\rho.\beta(t) - \\ & - 2.\eta_7.\alpha(t)^2 + \eta_6.\rho.\sigma(t) - \eta_5.\alpha(t).\sigma(t) - \eta_6.s[\sigma(t)]\} \end{aligned} \tag{27}$$

Equation (27) allows us to find the steady state relation

$$(28) \quad \overline{\alpha} = \frac{\eta_2'}{\eta_1'}.\rho - \frac{\eta_6}{\eta_1'}.s(\overline{\sigma})$$

which is similar to (21), for $\eta_1' = \overline{v}_\alpha$, $\eta_2' = \overline{v}_\beta$ and $\eta_6 = \overline{v}_{\beta\sigma}$. If one also assumes that $\eta_3' = \overline{v}_\sigma$, $\eta_4 = \overline{v}_{\alpha\beta}$, $\eta_5 = \overline{v}_{\alpha\sigma}$, $\eta_7 = 2.\overline{v}_{\alpha\alpha}$, $\eta_8 = 2.\overline{v}_{\beta\beta}$ and $\eta_9 = 2.\overline{v}_{\sigma\sigma}$, then (26) will correspond to a Taylor-series expansion of v around the steady state point $(\overline{\alpha}, \overline{\beta}, \overline{\sigma})$. Nevertheless, there are now differences in the results from considering the two different approaches to address the problem's stability. Now, the Jacobian matrix is different from (23), what necessarily implies a different set of stability conditions

$$(29) \quad J' = \begin{bmatrix} 0 & -\left[\frac{\overline{v}_\beta}{\overline{v}_\alpha}.\rho - \frac{\overline{v}_{\beta\sigma}}{\overline{v}_\alpha}.s(\overline{\sigma})\right] \\ \omega' & \rho \end{bmatrix}, \text{ with}$$

$$\omega'' = \frac{1}{v_{\beta\beta}}.\left[\left(\overline{v}_{\alpha\beta} - \frac{\overline{v}_\beta}{\overline{v}_\alpha}.\overline{v}_{\alpha\alpha}\right).\rho - \overline{v}_\alpha + \frac{\overline{v}_{\beta\sigma}}{\overline{v}_\alpha}.\overline{v}_{\alpha\alpha}.s(\overline{\sigma})\right]$$

Comparing (29) with (23), we see that now full stability in the steady state vicinity is not possible, because the sum of the Jacobian matrix eigenvalues is always a positive quantity. Saddle-path stability is found if one of the following set of conditions apply [together with (22)],

$$(30) \quad \begin{aligned} &i)\ \overline{v}_{\beta\beta} > 0;\ \rho < \frac{\overline{v}^2_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta\sigma}.s(\overline{\sigma})}{\overline{v}_{\alpha\beta}.\overline{v}_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_\beta} \\ &ii)\ \overline{v}_{\beta\beta} < 0;\ \rho > \frac{\overline{v}^2_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_{\beta\sigma}.s(\overline{\sigma})}{\overline{v}_{\alpha\beta}.\overline{v}_\alpha - \overline{v}_{\alpha\alpha}.\overline{v}_\beta} \end{aligned}$$

Note that in the presence of the exogenous variable, when conditions (25) or (30) are satisfied, the slope of the stable arm continues to be positive and a same kind of exogenous disturbances analysis as in section 4, with similar results, can be conducted. Since the analysis leads to a same set of qualitative results we do not proceed with it further.

We can summarize this section's results, by stating that:

a) If the exogenous variable that was introduced in the OGC model is not a constant value in the steady state, then stability conditions differ from the result in expression (11);

b) Under the new scenario, there is an important difference between the analysis of the problem using a linear quadratic approximation of the objective function or, in alternative, proceeding with the computation of the optimality conditions maintaining the objective function in its generic form. This difference is specially important if we take into account that one of the procedures leads to the possibility of a stable node (two negative eigenvalues associated with the Jacobian matrix) while the other only accounts for the possibility of saddle-path stability (one negative eigenvalue);

c) Although the second-order approximation of the objective function simplifies computation of optimality conditions and of the Jacobian matrix, it ends up by ignoring a set of terms that obscure the true conditions under which stability is observed.

6. Illustration with a Fertility – Human Capital Case

The previous structure can be applied to several types of economic problems. In Gomes (2004*a*; 2004*b*), a technology framework was considered and the assumed growth rates were the growth rate of applied knowledge and the growth rate of a science frontier. To illustrate our OGC model's dynamics, we consider a fertility-human capital trade-off, in line with the discussion in Becker and Barro (1988) and Becker et al. (1990). We will make use of the framework of sections 2 to 4; the introduction of exogenous variables in the objective function will not be considered.

Our main goal in this application consists in giving economic meaning to the variables used in the previous analysis. The economic meaning we pursue should have some compatibility with the evidence about population dynamics. We will reject the Malthusian view that there is a direct correlation between rising prosperity and population growth and rely on the evidence that points to the following facts (Barro and Sala-i-Martin, 1995; Barro and Lee, 1993, 2000):

i) increases in per capita income tend to reduce fertility, except for very poor countries or social groups. This evidence leads to the idea that high income households tend to give more attention (and attribute higher utility) to the education of children than to the number of children;

ii) households are altruistic, that is, parents are concerned with the well being of their children and therefore they are interested in investing in human capital (in the education of their children);

iii) There is a conflict between the quantity and the quality of children; households would like to have more children and better prepared children, but one of the goals conflicts with the other, given some resource constraint.

Let variable $B(t)$ represent the number of children that a given set of households may have. The fertility rate, $\beta(t)$, is a control variable, therefore it is possible to choose how many children to have, given the two following objectives: individuals withdraw utility from a higher number of children ($v_\beta > 0$), but they also want the young generation to have access to a good education. Letting $A(t)$ be a human capital variable, $\alpha(t) \equiv A(t)/B(t)$ reflects the concern of having the highest human capital level per child if the following objective function derivative is verified: $v_\alpha > 0$. Note, in our model, that the growth rate of human capital is an exogenous constant value, a.

It makes sense, from an economic point of view, to assume decreasing marginal utility [Becker and Barro (1988) also takes this assumption]: additional children have a positive effect in the households utility but the effect tends to decrease; similarly, higher human capital levels per child have an higher impact over the households utility for small amounts of already accumulated human capital. Therefore, $v_{\alpha\alpha} < 0$ and $v_{\beta\beta} < 0$; also, $v_{\alpha\beta} > 0$ is likely to hold.

As far as the resource constraint is concerned, equation (2) translates the referred trade-off between the human capital–fertility ratio growth rate and the fertility rate. For a given level of income (considered constant), resources have to be distributed between more children and better prepared children.

In the previous paragraphs we have only given economic meaning to the set of variables of the general OGC model developed along the previous sections of the paper. Besides this, the definition of variables also allowed us to attribute signs to the derivatives of the objective function. Some conclusions are worthwhile to mention now:

(*i*) Under the model's assumptions, in the steady state the economy's fertility rate is also the rate at which human capital grows;

(*ii*) According to (11), saddle-path stability requires a discount rate for future utility above a certain positive level. Note that instability would mean the existence of a very high human capital level but a tendency for the fertility rate to go to zero (this is apparently the unstable result that developed countries experience nowadays); or, alternatively, an increasing fertility rate accompanied by a tendency of human capital to diverge towards very low levels (somehow, what the developing countries have been experiencing). To make it possible to converge to a steady state where optimal levels of human capital and optimal fertility rates are accomplished, it is necessary to have a strong discount of future outcomes.

(*iii*) The stable trajectory (12) is positively sloped which means that the qualitative evolution of the two variables is identical: to the left of the steady state locus, both variables (the fertility growth rate and the human capital-population ratio) have their values evolving positively towards the steady state point. To the right of the steady state, the decline of the ratio value is accompanied with a decrease of the fertility growth rate.

(*iv*) It is also interesting to show what happens when some public policy or efficiency gain in the private economy stimulates the growth of human capital. A higher growth rate a implies, as verified previously in the general case, that the saddle-path shifts to the left, implying an immediate jump in the equilibrium solution – ratio $\alpha(t)$ is not changed, but fertility grows faster. Thus, under our model's assumptions, the stimulus to education means that individuals may have more children, without losing the possibility to give to their descendents the same level of education.

(*v*) Finally, as in the general case, we observe that a higher discount rate of future utility allows for a long run higher α ratio, without any loss in the fertility growth rate. Nevertheless, in the short-run there is effectively a loss in this rate value.

7. Final Remarks

This paper has served two purposes. First, it allows us to compare, for a simple dynamic model, the virtues and flaws of considering a linear quadratic approximation of the objective function around the steady state point, before finding optimality conditions. Despite recent literature pointing to the importance of such a method, namely under stochastic scenarios, we have found that in a purely deterministic setup, the second-order approximation may lead to inaccurate results regarding stability conditions if one considers exogenous variables as arguments of the objective function.

Second, a particular class of growth models, that we have named OGC models, was thoroughly analysed and we were able to find, under a general framework and using two different techniques, conditions under which saddle-path stability holds. These conditions impose bounds to the discount rate through which future achievements are valuated.

The proposed framework can be taken as a general setup that can be adapted to several kinds of problems. To exemplify the applicability of the OGC problem, we have considered a fertility--human capital example. The economic meaning of the trade-off between having many children and having less children but more well educated to enter the labor market can be analysed under our framework. In particular we found that this problem may lead to an unstable outcome or, under a particular condition, to saddle-path stability. If stability prevails, the system converges to a steady state point characterized by a positive, constant and identical growth rate for fertility and human capital.

References

Barro, Robert J.; Lee, Jong-Wha (1993) International Comparisons of Educational Attainment, *Journal of Monetary Economics*, 32, 363-394.

Barro, Robert J.; Lee, Jong-Wha (2000) International Data on Educational Attainment Updates and Implications, *Working paper*, Harvard University.

Barro, Robert J.; Sala-i-Martin, Xavier (1995) *Economic Growth*, New York, McGraw-Hill.

Becker, Gary S.; Barro, Robert J. (1988) A Reformulation of the Economic Theory of Fertility, *Quarterly Journal of Economics*, 103, 1-25.

Becker, Gary S. et al. (1990) Human Capital, Fertility and Economic Growth, *Journal of Political Economy*, 98, S12-S37.

Benigno, Pierpaolo; Woodford, Michael (2004) Optimal Monetary and Fiscal Policy: A Linear-Quadratic Approach, *International Finance Discussion Papers*, number 806, Board of Governors of the Federal Reserve System.

Benveniste, Lawrence M.; Scheinkman, José A. (1982) Duality Theory for Dynamic Optimization Models of Economics: the Continuous Time Case, *Journal of Economic Theory*, 27, 1-19.

Collard, Fabrice; Juillard, Michel (2001) Perturbation Methods for Rational Expectations Models, Paris, *CEPREMAP working paper*.

Díaz-Giménez, Javier (1998) Linear Quadratic Approximations: an Introduction, in R. Marimon and A. Scott (eds.), *Computational Methods for the Study of Dynamic Economies*, Oxford University Press, chapter 2.

Gomes, Orlando (2004*a*) The Optimal Control of Technology Choices, in M. A. Ferreira, R. Menezes and F. Catanas (eds.), *Themes in Quantitative Methods 4*, Lisbon, Edições Sílabo, 255-273.

Gomes, Orlando (2004*b*) A Second-Order Approximation to Technology Choices, EconWPA working paper (http://econwpa.wustl.edu/), September, (JEL: C6, D5, D9) 0409007.pdf.

Judd, Kenneth (1998) *Numerical Methods in Economics*, Cambridge MA, MIT Press.

Pitchford, John D. (1977) *Applications of Control Theory to Economic Analysis*, Amsterdam, North-Holland.

Schmitt-Grohé, Stephanie; Uribe, Martin (2002) Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function, *NBER working paper* number 0282.

Sims, Christopher (2000) *Second-Order Accurate Solutions of Discrete Time Dynamic Equilibrium Models*, unpublished manuscript, Princeton University.

Sutherland, Alan (2002) A Simple Second-Order Solution Method for Dynamic General Equilibrium Models, *University of St. Andrews discussion paper* number 0211.

Woodford, Michael (2001) Inflation Stabilization and Welfare, *NBER working paper* number 8071.

Woodford, Michael (2003) *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton, New Jersey, Princeton University Press.